Inverse scattering transform for the Toda lattice with steplike initial data

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Abstract. We study the solution of the Toda lattice Cauchy problem with steplike initial data. The initial data are supposed to tend to zero as $n \to +\infty$. By the inverse scattering transform method formulas allowing us to find solution of the Toda lattice is obtained.

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1. Introduction

The Toda lattice has some very important applications in the theory of physics of nonlinear processes (see [1]). It is known the inverse scattering method allows one to investigate in detail the Cauchy problem for the Toda lattice in the different classes of initial data (see [1]-[15] and references therein). The last problem for the doubly-infinite Toda lattice

$$\begin{cases} \dot{a}_n = \frac{a_n}{2}\left(b_{n+1} - b_n\right), \quad \cdot = \frac{d}{dt}, a_n = a_n(t) > 0, \\ \dot{b}_n = a_n^2 - a_{n-1}^2, \quad b_n = b_n(t), \;\; n = 0, \pm 1, \pm 2, ... \end{cases} \tag{1.1}$$

with fast stabilized or steplike fast stabilized initial data is investigated in [1]-[9] (see also references therein) by the method of inverse scattering transform. However, this problem is not studied in the case of steplike initial data, where a_n tend to zero as $n \to +\infty$ (or $n \to -\infty$).

In this paper we study the Cauchy problem for the system (1.1) with initial data

$$a_n(0) \to 0, \; b_n(0) \to 0 \quad as \quad n \to +\infty,$$

$$\sum_{n<0} |n| \left\{ |a_n(0) - 1| + |b_n(0)| \right\} < \infty \; . \tag{1.2}$$

The solution is considered in the class

$$\|a_n(t)\|_{C[0,T]} \to 0, \; \|b_n(t)\|_{C[0,T]} \to 0, \quad \texttt{as} \;\; n \to +\infty,$$

$$\|Q(t)\|_{C[0,\,T]} < \infty \; , \tag{1.3}$$

for arbitrary $T > 0$, where

$$Q(t) = \sum_{n<0} |n| \left(|a_n(t) - 1| + |b_n(t)| \right) \; .$$

Note, we cannot apply directly method given in [1]-[9] for the case $\inf_n a_n > 0$, because the Jost solution with the asymptotic behaviour on an $+\infty$ does not exist in our case. On the other hand, method of inverse problem is used (see [10]) in the case when Jacobi operator associated with (1.1) has the continuous spectrum $[a, \; b]$ of multiplicity two. But this method cannot be used when the spectrum of the Jacobi operator has a continuous spectrum of multiplicity one and a discrete spectrum.

The paper is organized as follows. In section 2 we formulate some auxiliary facts to the inverse scattering problem for the Jacobi operator associated with (1.1)-(1.2). In section 3 we describe the evolution of the scattering data of problem (1.1)-(1.2) .

In the last section we prove existence of the solution of the problem (1.1)-(1.2) in class (1.3).

2. The scattering problem

Consider Jacobi operator L generated in $\ell^2(-\infty, \infty)$ by the finite-difference operations

$$(Ly)_n = a_{n-1}y_{n-1} + b_n y_n + b_n y_{n+1},$$

in which the real coefficients $a_n > 0,\ b_n$ satisfy the conditions

$$a_n \to 0,\ b_n \to 0 \quad as \quad n \to +\infty,$$

$$\sum_{n<0} |n| \{|a_n - 1| + |b_n|\} < \infty \ .$$

The interval $[-2,\ 2]$ is the continuous spectrum of multiplicity one of operator L (see [16],[17]). Beyond the continuous spectrum, L can have a finite number of simple eigenvalues $\mu_k(t),\ k = 1, ..., p$.

Let us formulate some auxiliary facts related to the inverse scattering problem for the equation

$$(Ly)_n = \lambda y_n, \quad n = 0,\ \pm 1, ...,\ \ \lambda \in \mathbf{C} \tag{2.1}$$

Many of these facts can be found in [16],[17].

Let $P_n(\lambda)$ and $Q_n(\lambda)$ be solutions of Eq. (2.1) with initial conditions

$$P_{-1}(\lambda) = 0,\ P_0(\lambda) = 1,$$

$$Q_0(\lambda) = 0,\ Q_1(\lambda) = \frac{1}{a_0}.$$

We denote by L_0 semi-infinite Jacobi operator generated $\ell^2[0,\ \infty)$ by Eq. (2.1) as $n \geq 0$ and the boundary condition $y_{-1} = 0$. This operator is completely continuous. Moreover, the spectral function $\rho(\lambda)$ of L_0 represented [18] in the form

$$\rho(\lambda) = \sum_{\lambda_n < \lambda} \beta_n^{-2}$$

where λ_n is the eigenvalue of L_0 and β_n is the norm of the eigenfunction corresponding to the λ_n.

As is known from [18]-[19], the right Weyl function of the problem (2.1) has the form

$$m(\lambda) = \int_{-\infty}^{\infty} \frac{d\rho(\tau)}{\tau - \lambda}, \tag{2.2}$$

or

$$m(\lambda) = \sum_{n=1}^{\infty} \frac{\beta_n^{-2}}{\lambda_n - \lambda}.$$

where $\lambda_n \to 0$ as $n \to \infty$. It follows from [12]-[13] that for $\lambda \neq \lambda_k,\quad k = 1, 2, ...,$ Eq.(2.1) has Weyl solution

$$\psi_n(\lambda) = \ Q_n(\lambda) + m(\lambda)P_n(\lambda), \tag{2.3}$$

"on the right semiaxis" (such that $\sum_{n=0}^{\infty} |\psi_n(\lambda)|^2 < \infty.$).

Suppose that Γ is the complex λ–plane with cut along the interval $[-2, 2]$. In the plane Γ, consider the function

$$z(\lambda) = \frac{\lambda}{2} + \sqrt{\frac{\lambda^2}{4} - 1}$$

choosing the regular branch of the radical so that $\sqrt{\frac{\lambda^2}{4} - 1} < 0$ for $\lambda > 2$. We often omit the dependence of $z(\lambda)$ on λ in what follows. Thus, in the formulas involving z and λ, we always assume that z is as in the above equation.

It is well known (see, for example, [20]) that Eq.(2.1) has a Jost solution represented in the form

$$f_n(\lambda) = \alpha_n z^{-n}\left(1 + \sum_{m<0} A_{nm} z^{-m}\right). \tag{2.4}$$

The coefficients are given by

$$a_n = \frac{\alpha_n}{\alpha_{n+1}}, \quad b_n = A_{n,-1} - A_{n+1,-1}. \tag{2.5}$$

Without restriction of generality we can suppose that $\lambda_m \in (-2,\ 2)$ for any $m = 1, 2, ...$ As known [16],[17], for $\lambda \in \partial\Gamma,\ \ \lambda^2 \neq 4,\ \ \lambda \neq \lambda_m$ identity

$$\psi_n(\lambda) = a(\lambda)\overline{f_n(\lambda)} + \overline{a(\lambda)} f_n(\lambda) \tag{2.6}$$

holds, where the function $a(\lambda)$ can be regularly continued to Γ. Note also, $a(\lambda)$ can have a finite number of coinciding simple zeros outside the interval $[-2,\ 2]$, because, these zeros constitute the discrete spectrum $\mu_k,\ \ k = 1, ..., p$, of the operator L.

Introduce reflection $R(\lambda)$ coefficient by the formula

$$R(\lambda) = \frac{\overline{a(\lambda)}}{a(\lambda)}.$$

The function $R(\lambda$ is continuous for $\lambda \in \partial\Gamma$. Setting $n = -1$ and $n = 0$ in the identity (2.6) yields the expression

$$m(\lambda) = -\frac{1}{a_{-1}} \frac{\overline{f_0(\lambda)} + R(\lambda) f_0(\lambda)}{f_{-1}(\lambda) + R(\lambda) f_{-1}(\lambda)} \tag{2.7}$$

The norming constants $M_k(t)$ corresponding to the $\mu_k(t)$ are given as

$$M_k^{-2} = \sum_{n=-\infty}^{\infty} f_n^2(\mu_k)\ ,\ \ k = 1, ..., p.$$

The set of quantities $\{R(\lambda); \mu_k;\ \ M_k,\ \ k = 1, ..., p\}$ is called the scattering data for the Jacobi operator L. The inverse scattering problem for L is to recover the coefficients $a_n,\ b_n$ from the scattering data.

In solving the inverse problem, an important role is played by the Marchenko-type basic equation. Define

$$F_n = \sum_{k=1}^{p} M_k^{-2} z_k^{-n} + \frac{1}{2\pi i}\int_{\partial\Gamma} \frac{R(\lambda)}{z^{-1}-z} z^{-n} d\lambda, \tag{2.8}$$

where $z_k = z(\mu_k),\ k = 1, ..., p.$

Then A_{nm} and α_n involved in (2.4) satisfy the relations

$$F_{2n+m} + A_{nm} + \sum_{k<0} A_{nk} F_{2n+m+k} = 0, \quad m < n \leq 0\ , \tag{2.9}$$

$$\alpha_n^{-2} = 1 + F_{2n} + \sum_{k<0} A_{nk} F_{2n+k}, \quad n \leq 0\ . \tag{2.10}$$

To reconstruct the operator L, we consider Eq.(2.8) which is constructed by the scattering data. We find A_{nm} and α_n from Eqs.(2.9) and (2.10), respectively, the firs one having a unique solution with respect to A_{nm}.The coefficients a_n *and* b_n are definides for $n < 0$ by (2.5). $f_n(\lambda)$ for $n \leq 0$ are defined by (2.4). From the formula (2.7) we obtain Weyl function $m(\lambda)$. The spectral measure $d\rho(\lambda)$ can be found by the formula

$$d\rho(\lambda_n) = \lim_{\lambda\to\lambda_n} (\lambda_n - \lambda) m(\lambda),\ \ n = 1, 2, ...$$

Using the approach in [12],[13],[19], we can reconstruct semi-infinite Jacobi operator L_0 by its spectral measure $d\rho(\lambda)$. Therefore, we find $a_n,\ b_n$ for $n \geq 0$.

3. Evolution of the scattering data

In this section we use the inverse scattering transform method to solve the problem (1.1)-(1.2). Let $a_n(t),\ b_n(t)$ be a solution of the problem (1.1)-(1.2) satisfying (1.3). Consider the Jacobi operator $L = L(t)$ associated with $a_n = a_n(t),\ b_n = b_n(t)$. Jost and Weyl solutions, reflection coefficient, spectral measure now depend on the additional parameter $t \in [0,\ \infty)$.

Theorem 1. *If the coefficients $a_n = a_n(t),\ b_n = b_n(t)$ of Eq.(2.1) are solutions to problem (1.1)-(1.2) in the class (1.3), then the evolution of the scattering data is described by the formulas*

$$R(\lambda, t) = R(\lambda, 0)\, e^{\left(z^{-1}-z\right)t} \tag{3.1}$$

$$\mu_k(t) = \mu_k(0),\ \ k = 1, ..., p \tag{3.2}$$

$$M_k^{-2}(t) = M_k^{-2}(0) e^{\left(z_k^{-1}-z_k\right)t}, \qquad z_k = z(\mu_k)\ ,\ \ k = 1, ..., p. \tag{3.3}$$

Proof. System (1.1) is represented (see, for example [8],[13]) in the Lax form

$$\dot{L} = [L,\ A] = AL - LA, \tag{3.4}$$

where $A = A(t)$ are Jacobi operator in $\ell^2\,(-\infty, \infty)$:

$$(Ay)_n = \frac{1}{2} a_n y_{n+1} - \frac{1}{2} a_{n-1} y_{n-1}.$$

Since (3.4) implies that the family of operators $L = L(t)$ are unitarily equivalent (see [5],[8]), the spectrum of $L = L(t)$ does not depend on n and (3.2) is valid.

Let $f_n(\lambda,\ t)$and $\psi_n(\lambda,\ t)$respectively be the Jost and Weyl solutions of the Eq.(2.1) with the parameter t. Consider the indetity (2.6) with the parameter t. As follows from [8],[12] the function $\frac{d}{dt}\psi_n - (A\psi)_n$ is also a solution of the Eq.(2.1) with the parameter t. Appling the operator $\frac{d}{dt} - A$ to (2.6), taking into account that the Jost solution $f_n(\lambda,\ t)$ does not depend (see [8], on t asymptotically, we obtain

$$\begin{aligned}\tfrac{d}{dt}\psi_n - (A\psi)_n &= \left(\dot a(\lambda,\ t) + \tfrac{1}{2}\left(z^{-1} - z\right) a(\lambda,\ t)\right) \overline{f_n(\lambda,\ t)} + \\ &+ \left(\overline{\dot a(\lambda,\ t)} - \tfrac{1}{2}\left(z^{-1} - z\right) \overline{a(\lambda,\ t)}\right) f_n(\lambda,\ t).\end{aligned} \tag{3.5}$$

On the other hand, we find

$$\frac{d}{dt}P_0 - (AP)_0 = \frac{b_0 - \lambda}{2}, \quad \frac{d}{dt}P_{-1} - (AP)_{-1} = -a_{-1}\ .$$

Since $P_n(\lambda,\ t)$ and $Q_n(\lambda,\ t)$ are linearly independent, the function $\frac{d}{dt}P_n - (AP)_n$ can be represented as

$$\frac{d}{dt}P_n - (AP)_n = A(\lambda,\ t)P_n + D(\lambda,\ t)Q_n.$$

Setting $n = -1$ and $n = 0$ in the last relation, we find that

$$A(\lambda,\ t) = \frac{b_0 - \lambda}{2}, \quad D(\lambda,\ t) = a_{-1}^2.$$

Therefore,

$$\frac{d}{dt}P_n - (AP)_n = \frac{b_0 - \lambda}{2}P_n + a_{-1}^2 Q_n.$$

The same arguments are valid for solution $Q_n(\lambda,\ t)$. Thus, we have the formula

$$\frac{d}{dt}Q_n - (AQ)_n = -P_n + \frac{\lambda - b_0}{2a_0}Q_n.$$

Now by the formula (2.3) with the parameter t we find that

$$\begin{aligned}\tfrac{d}{dt}\psi_n - (A\psi)_n &= \left(a_{-1}^2 m(\lambda,\ t) + \tfrac{\lambda - b_0}{2a_0}\right) Q_n + \\ &+ \left(\dot m(\lambda,\ t) + \tfrac{b_0 - \lambda}{2} m(\lambda,\ t) - 1\right) P_n.\end{aligned} \tag{3.6}$$

Since $L = L(t)$ is selfadjoint and bounded, $\frac{d}{dt}\psi_n - (A\psi)_n$ must satisfy the relation

$$\frac{d}{dt}\psi_n - (A\psi)_n = \theta(\lambda,\ t)\psi_n. \tag{3.7}$$

Hence, we can represent the function $\frac{d}{dt}\psi_n - (A\psi)_n$ as

$$\frac{d}{dt}\psi_n - (A\psi)_n = \theta(\lambda,\ t)Q_n + \theta(\lambda,\ t)m(\lambda,\ t)P_n. \tag{3.8}$$

Comparing this identity with (3.6), we have

$$\theta(\lambda,\ t) = a_{-1}^2 m(\lambda,\ t) + \frac{\lambda - b_0}{2a_0}, \tag{3.9}$$

Further, according to (2.6), (3.5), (3.7),

$$\theta(\lambda,\ t)a(\lambda,\ t)\overline{f_n} + \theta(\lambda,\ t)\overline{a(\lambda,\ t)}f_n = \left(\dot{a}(\lambda,\ t) + \tfrac{1}{2}\left(z^{-1} - z\right)a(\lambda,\ t)\right)\overline{f_n} +$$

$$+\left(\overline{\dot{a}(\lambda,\ t)} - \tfrac{1}{2}\left(z^{-1} - z\right)\overline{a(\lambda,\ t)}\right)f_n.$$

Since f_n and $\overline{f_n}$ are linearly independent, so substituting (3.9) into the last indetity, we obtain

$$\dot{a}(\lambda,\ t) + \tfrac{1}{2}\left(z^{-1} - z\right)a(\lambda,\ t) = \left(a_{-1}^2 m(\lambda,\ t) + \tfrac{\lambda - b_0}{2a_0}\right)a(\lambda,\ t),$$

$$\overline{\dot{a}(\lambda,\ t)} - \tfrac{1}{2}\left(z^{-1} - z\right)\overline{a(\lambda,\ t)} = \left(a_{-1}^2 m(\lambda,\ t) + \tfrac{\lambda - b_0}{2a_0}\right)\overline{a(\lambda,\ t)}.$$

From this relations, we get

$$\dot{R}(\lambda, t) = (z^{-1} - z)R(\lambda, t),$$

which imply (3.1).

Now, let $g_n(\mu_k,\ t)$ be a normalized eigenfunction of L. Since the eigenvalues $\mu_k, k = 1, ..., p$, of this operator are simple, we have

$$\frac{d}{dt}g_n - (Ag)_n = cg_n.$$

Taking the scalar products of g_n with both sides of this equality in $\ell^2(-\infty, \infty)$ and using $\|\psi_n\|_{\ell^2(-\infty,\infty)} = 1$ and $A^* = -A$, we obtain $c = 0$. Therefore,

$$\frac{d}{dt}g_n - (Ag)_n = 0 \tag{3.10}$$

On the other hand, if a normalized eigenfunction $g_n(\mu_k,\ t)$ corresponds to the eigenvalue μ_k, then

$$g_n(\mu_k,\ t) = c_k(t)f_n(\mu_k,\ t).$$

This implies that $M_k^2(t) = c_k^2(t)$. By virtue of (2.4), we find that

$$\frac{d}{dt}g_n - (Ag)_n \sim \left(\dot{c}_k(t) + \frac{z_k - z_k^{-1}}{2}c_k(t)\right)z_k^{-n}$$

as $n \to -\infty$. Taking into account (3.10), we have

$$\dot{c}_k(t) + \frac{z_k - z_k^{-1}}{2}c_k(t) = 0$$

This equation implies the relation (3.3).

The theorem is proved.

Using Theorem 1, we obtain the following procedure for solving problem (1.1),(1.2) based on the inverse scattering transform method: Initial data (1.2) is given. Construct $R(\lambda, 0)$,$\mu_k(0)$, $M_k(0)$, $k = 1, ..., p$. Calcullate $R(\lambda, t)$,$\mu_k(t)$,$M_k(t)$ using formulas (3.1)-(3.3). Construct a solution by solving the inverse problem by applying approach of the section 2 with $R(\lambda, 0)$,$\mu_k(0)$, $M_k(0)$, $k = 1, ..., p$ replaced by (3.1)-(3.3).

4. Solvability of the Cauchy problem for the Toda lattice

In section 3, while constructing a solution to problem (1.1)-(1.2), we assumed that this solution exists in the class (1.3). Let us now investigate its existence.

Theorem 2. *The problem (1.1)-(1.2) has a unique solution in the class (1.3).*

Proof. Denote by B the Banach space of pairs of sequences $y = (y_{1,n},\ y_{2,n})_{n=-\infty}^{\infty}$ for which the norm $\|y\|_B = \sup\limits_{n\geq 0}(|y_{1,n}| + |y_{2,n}|) + \sum\limits_{n<0}|n|(|y_{1,n}| + |y_{2,n}|)$ is finite. Then (see [21]) the set $C([0,\ T];\ B)$ of the continuous on an interval $[0,\ T]$ with respect to the norm $\|\cdot\|_B$ functions is the Banach space.

Let as assume that

$$x_{1,n} = \begin{cases} a_n(t) \quad for \quad n \geq 0, \\ \\ a_n(t) - 1 \quad for \ \ n < 0, \end{cases} \tag{4.1}$$

$$x_{2,n} = b_n(t).$$

Then system (1.1) is equivalent to the system

$$\begin{cases} \dot{x}_{1,n} = \frac{1}{2}x_{1,n}\left(x_{2,n+1} - x_{2,n}\right) + \frac{1}{2}\left(1 - \delta_{n,|n|}\right)\left(x_{2,n+1} - x_{2,n}\right), \\ \\ \dot{x}_{2,n} = x_{1,n}^2 - x_{1,n-1}^2 + 2\left(1 - \delta_{n,|n|}\right)\left(x_{1,n} - x_{2,n-1}\right), \end{cases} \tag{4.2}$$

where $\delta_{n,m}$ is the Kronecker symbol.

Denote by F the operator generated the right-hand sides of system (4.2). Note, operator F is strongly continuously differentable in the space $C([0,\ T];\ B)$.

Now passing to the integral equation in the standard manner, we find problem (4.2) with initial conditions

$$x_{1,n}(0) = \begin{cases} a_n(0) \quad for \quad n \geq 0, \\ \\ a_n(0) - 1 \quad for \ \ n < 0, \end{cases} \tag{4.3}$$

$$x_{2,n}(0) = b_n(0).$$

is equivalent to the equation

$$x(t) = x(0) + \int_0^t F(x(\tau))d\tau \tag{4.4}$$

Applying the principe of compressed maps, we find that problem (4.4) on some interval $[0,\ \delta]$ has a unique solution $x(t)$with finite norm $\|x(t)\|_{C([0,\ \delta];\ B)} < \infty$. Let us show that this solution can be extended to the entire positive semiaxsis. Assume the opposite. Then there exists a point $t^* \in (0,\ \infty)$such that problem (4.2)-(4.3) has a solution $x(t) = (x_{1,n}(t),\ x_{2,n}(t))$on the interval $[0,\ t^*)$ but $\overline{\lim\limits_{t\to t^*-0}}\|x(t)\|_B = \infty$. It follows from [8],[13] problem (1.1)-(1.2) has a unique solution$(a_n(t),\ b_n(t))$ in $C^\infty([0,\ \infty);\ M)$,

where $M = \ell^{\infty}(-\infty, \infty) \oplus \ell^{\infty}(-\infty, \infty)$. Hence, according to the (4.1) problem (4.2)-(4.3) has a unique solution $x(t) = (x_{1,n}(t),\ x_{2,n}(t))$ satisfying

$$|x_{1,n}(t)| + |\,x_{2,n}(t)| < C$$

for any $t \in [0,\ \infty)$, where C does not depend on t . We integrate the system (4.2) over a interval $[0,\ t]$. Then, using the last inequality, after some simple transformations , we get

$$\|x(t)\|_B \leq 2\,\|x(0)\|_B + (4C+4)\int_0^t \|x(\tau)\|_B\, d\tau,\ \ 0 < t < t^*\ ,$$

which, according to the Gronwall's inequality implies

$$\|x(t)\|_B \leq 2\,\|x(0)\|_B\, e^{(4C+4)t}.$$

Therefore, our assumption that $\overline{\lim\limits_{t\to t^*-0}}\, \|x(t)\|_B = \infty$ is not correct and problem (4.2)-(4.3) has a unique solution $x(t) = (x_{1,n}(t),\ x_{2,n}(t)) \in C\left([0,\ T]\,;\ B\right)$ for any $T > 0$. Integrating the system (1.1) over a interval $[0,\ t]$ and using (4.1), we obtain that problem (1.1)-(1.2) be uniquely solvable in the class (1.3).

Thus, the theorem is proved.

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